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# **When Does Immigration Facilitate Efficiency ?**

**A.W.Anwar\***  
**Department of Economics**  
**Edinburgh University**  
**William Robertson Building**  
**50 George Square**  
**Edinburgh**  
**EH8 9JY**  
**e-mail: Ahmed.Anwar@ed.ac.uk**

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## **Abstract**

This paper adds to the growing literature on stochastic evolutionary models. These models can be characterised by small probability shocks or mutations which perturb the system away from its deterministic evolution, allowing it to move between equilibria over a long period of time. Much of the literature has concentrated on the result that, in the limit as the mutation rate approaches zero, the stationary distribution becomes concentrated on the risk-dominant equilibrium because it is easier to flow into. However, it has been shown that in models of local interaction, allowing player movement eases the flow into the efficient equilibrium. This paper looks at the consequences of such player movement when there are capacity constraints which limit the number of agents who can reside at each location. The system may then settle into a mixed state in which different locations coordinate on different equilibria.

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## 1. Introduction

How do players know which equilibrium to play when a game has multiple equilibria? This question has been at the heart of much research in game theory. The focus of attention has been the 2x2 Coordination Game such as the one given in figure 1.1 that has two Nash equilibria in pure strategies, one of which is Pareto-efficient but is riskier to play than the other. Harsanyi and Selten (1988) call the former equilibrium payoff-dominant and the latter risk-dominant. Schelling (1960) appeals to the prominence of efficiency to suggest that agents will play for the payoff-dominant equilibrium in the expectation that other agents will be similarly attracted by its focal status. But Harsanyi and Selten have emphasised that such an expectation may not be well-founded. If each player optimises on the assumption that the opponent is equally likely to play either strategy, the outcome will be the risk-dominant equilibrium of the game, which therefore also has a focal status that may outweigh that of the payoff-dominant equilibrium.

Evolutionary game theory has given the argument another perspective. By modelling the process by which agents adjust their strategies out of equilibrium we can analyse how it is that one equilibrium strategy rather than another may be selected. The principle underlying the dynamic systems studied in evolutionary game theory is that successful strategies will be used by a greater proportion of the population in future periods.

To illustrate the idea consider the Coordination Game of figure 1.1. The game has two pure-strategy equilibria,  $e_1 = (s_1, s_1)$  and  $e_2 = (s_2, s_2)$ . Notice that  $e_1$  is payoff-dominant while  $e_2$  is risk-dominant. There is also a mixed-strategy equilibrium where  $s_1$  is played with probability  $2/3$ . When expressed in terms of the fraction  $q$  of the population using strategy  $s_1$ , these Nash equilibria correspond respectively to  $q=1$ ,  $q=0$  and  $q=2/3$ . We begin by studying a

specific dynamic system for which the population states  $q=1$  and  $q=0$  are stable stationary points. Denote these stationary states by  $E_1$  and  $E_2$  respectively.

$s_1$	$s_2$	
$s_1$	5,5	0,3
$s_2$	3,0	4,4

Figure 1.1  
2x2 Coordination Game

Assume that members of the population are randomly matched each period to play this game. They adjust their choice by playing the strategy that yielded the highest expected payoff in the previous period when they are given the chance to do so. Now consider the case where  $q > 2/3$ . If a revision opportunity arises, then the optimal response against the current state is to play  $s_1$ . The proportion playing  $s_1$  will therefore grow over time until the state where everyone plays  $s_1$  is reached. The basin of attraction of  $E_1$  is therefore  $(2/3, 1]$ , since it will be selected from any state where  $q > 2/3$ . Similarly the basin of attraction of  $E_2$ , where everyone plays  $s_2$ , is  $[0, 2/3)$ . A third possible stationary state (provided the population size  $N$  is infinite) is  $q = 2/3$ . At this point, no agent has an incentive to change his strategy. However, only  $E_1$  and  $E_2$  are locally stable.

Kandori *et al* (1993) and Young (1993) added to this analysis by assuming that agents sometimes 'mutate' by changing their strategies at random. Each agent has a positive probability of mutating each period. There is therefore a small but positive probability that there will be a large number of simultaneous mutations. Once in an equilibrium, it is therefore no longer the case that the system will stay there forever because enough simultaneous mutations will eventually occur to move the system into the other basin of attraction. The system therefore

needs to be described in terms of a probability distribution over the states, with much of the time spent in or close to the two stable states when the mutation rate is small.

Kandori *et al* show that, when the probability of mutation goes to zero, the distribution becomes concentrated entirely on the risk-dominant equilibrium,  $E_2$ . The reason for this is that more mutations are required to move from  $E_2$  to  $E_1$  than from  $E_1$  to  $E_2$ . As the mutation rate goes to zero the probability of the first transition becomes negligible compared with the second. The time-limit of the distribution over population states therefore puts all its mass on  $E_2$  when the mutation rate becomes vanishingly small. Following Kandori *et al*, equilibria that have a positive probability as the mutation rate goes to zero will be called long-run equilibria.

A criticism of this model is the huge expected waiting time to move from  $E_1$  to the long-run equilibrium  $E_2$  when the population size is large. If the system is in  $E_1$  and the mutation rate is very small, then although it is true that the stationary distribution will be concentrated on  $E_2$ , it is likely to be a very long time before there are enough simultaneous mutations to move the system out of the basin of attraction of  $E_1$ . Ellison (1993) introduces a local interaction structure which dramatically reduces waiting times whilst maintaining the result that the state where everyone plays the risk-dominant strategy is the unique long-run equilibrium. In his model, players are located around a circle and interact only with a subset of the population who are close to them. He shows that a small number of mutations concentrated together may be enough to upset the payoff-dominant equilibrium<sup>1</sup>.

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<sup>1</sup> Suppose for example that each agent only interacts with the four closest players or neighbours either side of him and they are randomly matched with these neighbours to play the game in figure 1.1. Then each player has 8 neighbours and if at least 3 of them play  $s_2$  then the best response is to play  $s_2$ . Now consider the state in which everyone plays  $s_1$ . If there are 4 neighbours who simultaneously mutate then they each have 3 neighbours who are playing  $s_2$  and they will therefore continue to play  $s_2$ . There are now another 4 players who are currently playing  $s_1$  but have at least 3 neighbours playing  $s_2$ . They will revise their strategies when given the chance and in this way the strategy  $s_2$  will spread throughout the population. Four well placed mutations are enough to move the system into the basin of attraction of the risk-dominant equilibrium and so the expected waiting time is much smaller and independent of the population size.

Although the overwhelming consensus of this literature is that the risk-dominant equilibrium  $E_2$  will be selected, this is not always the case when local interaction is modelled. In Kandori *et al*, the location structure does not matter since each agent is equally likely to be matched with every other agent in the population. In models of local interaction, agents are more likely to be matched with neighbouring players. An agent's choice of location is therefore important, since this will determine his or her expected payoff. Thus, if agents are given the chance, they will move to the location where they get the highest expected payoff. In Ellison's model, however, this phenomenon is absent, since agents are located at *fixed* positions around a circle and remain there. If this assumption is relaxed, a few mutations need no longer be enough to upset the payoff-dominant equilibrium because agents may move away from a locality in which deviant mutations have occurred in search of a higher payoff. Similarly, the risk-dominant equilibrium may now be easier to upset since a few localised mutations may entice movement towards this locality. Ely (1995) presents a model based on this idea in which such movement makes the long-run equilibrium  $E_1$  rather than  $E_2$ .

However, movement between locations is sometimes restricted by capacity constraints that limit the number of people that can reside at each location. These constraints may arise from physical aspects of the locations or may be imposed by local governance. For example, there may be a certain number of *slots* at each location, limiting the number that can play there. Alternatively, one can imagine neighbouring societies who impose restrictions on the maximum number of inhabitants. Once this limit is reached, further immigration is prohibited. Such constraints may be imposed when there is a negative payoff to congestion.

We investigate the consequences of such constraints in a model where agents can move between two locations or *islands*. While a location is not full to capacity, agents can move freely to and from it. However, no one can move to a location that is full to its capacity.

We begin by analysing the case where strategy<sup>2</sup> revision is instantaneous, i.e. everybody revises their strategy each period, but the chance to move to another location only arises with some positive probability. It is shown that there is a range of parameter values for which the long-run equilibria involve one island playing the efficient equilibrium and the other playing the risk-dominant one with the first island full to capacity.

In section 2.2 we show that the results hold when there is inertia in strategy revision. Since one of the interpretations of the model is that capacity constraints may be imposed to limit congestion, we introduce a congestion effect in section 2.3 and show the results of the previous two sections still hold. Finally, section 2.4 looks at the consequences of relaxing the capacity constraints altogether. As in Ely (1995), the efficient equilibria are then favoured.

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<sup>2</sup> A strategy is simply a choice between  $s_1$  and  $s_2$  and does not include location.

## Section 2.

In this section, we present a model of local interaction with movement between locations. We assume that there are two islands and agents are randomly matched with someone on the same island to play the game of figure 2.1 in which  $A > C$ ,  $D > B$ ,  $A > D$  and  $A + B < C + D$ . Hence  $e_1 = (s_1, s_1)$  is the payoff-dominant equilibrium while  $e_2 = (s_2, s_2)$  is risk-dominant. The probability with which  $s_1$  is played in the mixed-strategy equilibrium is  $q^* = (D - B) / (A - C + D - B) > 1/2$ .

	$s_1$	$s_2$
$s_1$	A, A	B, C
$s_2$	C, B	D, D

Figure 2.1

Each period some agents are given the chance to move islands. We begin by looking at the case where strategy revision on each island is instantaneous. The analysis is then extended to the case where there is inertia in strategy revision. We then show that the results hold when there is a congestion effect. Finally, we look at the consequences of removing the capacity constraints.

### 2.1 Basic Model

Players are randomly matched on each of two isolated islands to play the game of figure 2.1. The global population is  $2N$  and the capacity of each island is  $(1+d)N$ . Strategy revision is instantaneous, that is everybody chooses a strategy that is a best response to the state in the previous period. The chance to change islands arises with a positive probability each period. When such an opportunity arises, the agent will choose the location and strategy that would have maximised their expected payoff in the previous period. If the agent is indifferent between two choices then we assume they choose either with a positive probability.



However, an agent cannot move to an island that is full to capacity. If the number of agents who wish to move to island  $i$  is greater than  $(N(1+d)-n_i)$ , where  $n_i$  is the current number on the island, then only  $(N(1+d)-n_i)$  of them will be allowed to move.  $N$  is sufficiently large that the following set of numbers are all integers,  $\{N(1-d), N(1+d), q^*N, (1+d)q^*N, (1-d)q^*N, (1-d)(1-q^*)N, (1+d)(1-q^*)N\}$ . One can think of the following story underlying these dynamics. At the end of each period players gather information on the proportion of the population using each strategy on their island. With some positive probability, they also learn the proportions on the other island. At the beginning of the next period they choose a location and strategy to use for that period. If they have no information about the other island then they stay where they are and choose the strategy that is a best reply to the proportions in the previous period. If they do learn the proportions on the other island then they will want to move if a best reply on the other island yields a higher expected payoff. If the island has spare capacity they will move and play the best reply. If it is full then they play a best reply on their current island.

The state space is

$$S = \left\{ \left( \frac{n_1^1}{n_1}, \frac{n_2^1}{2N - n_1} \right) : n_1^1 \in (0, 1, \dots, n_1), n_2^1 \in (0, 1, \dots, 2N - n_1), N(1-d) \leq n_1 \leq N(1+d) \right\},$$

where  $n_i^1$  is the number playing strategy  $s_1$  on island  $i$  and  $n_1$  is the number of agents on island

1. Denote a state of the system by  $s=(q_1, q_2, n_1) \in S$ , where  $q_i$  is the proportion of the population playing  $s_1$  on islands  $i$ .

The dynamics give rise to a Markov process,  $P$ , on state space  $S$ . From any initial condition, the system will move to a state or set of states where it remains. Following Young (1993), such a set will be called a recurrent communication class. The recurrent communication classes are characterised latter.

Without mutations, the system will move to one of these classes and remain there. Now assume that each agent mutates independently, with probability  $\epsilon$ , with the consequence

that a strategy<sup>3</sup> is re-selected at random on their current island. This allows the system to move between classes and gives rise to the perturbed transition matrix  $P^\epsilon$  where,

$$P_{ij}^\epsilon = P_{ij} (1 - \epsilon)^{2N} + \sum_{k=1}^{2N} c_{ijk} \epsilon^k (1 - \epsilon)^{2N-k}. \quad (2.1)$$

$P_{ij}$  is the  $ij$ th element of  $P$ , the unperturbed transition Matrix.

**Proposition 2.1:**  $P^\epsilon$  has a unique stationary distribution  $\pi(\epsilon)$  and  $\lim_{\epsilon \rightarrow 0} \pi(\epsilon)$  exists and is equal to one of the stationary distributions of  $P$ .

**Proof:** Young (1993) shows that this is true if  $P^\epsilon$  is a ‘regular perturbation’ of  $P$ . If  $P^\epsilon$  is a regular perturbation of  $P$  then the following conditions must hold,

- i)  $P^\epsilon$  is aperiodic and irreducible
- ii)  $\lim_{\epsilon \rightarrow 0} P_{ij}^\epsilon = P_{ij}$
- iii)  $P_{ij}^\epsilon > 0$  for some  $\epsilon$  implies  $\exists r \geq 0$  s.t.  $0 < \lim_{\epsilon \rightarrow 0} e^{-r} P_{ij}^\epsilon < \infty$ .

From (2.1) conditions (ii) and (iii) are clearly satisfied. If  $P_{ij}^\epsilon > 0$  then  $r$  is 0 if  $P_{ij} > 0$  or equal to the lowest value of  $k$  such that  $c_{ijk} > 0$ . We now show that  $P^\epsilon$  is aperiodic and irreducible. The diagonal elements of  $P^\epsilon$  are all positive. This is because in any state, there is a positive probability that nobody moves and that there are mutations that keep the same numbers playing each strategy on both islands. Hence  $P^\epsilon$  is aperiodic. There is a positive probability of going from any state to the class in which both islands coordinate on the same equilibrium. This simply requires a certain number of mutations on each island. We can then

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<sup>3</sup> All the results go through if we assume a strategy and location is re-selected at random with the restriction that the capacity constraint cannot be broken. If the number who re-select a location at random would take that location over its capacity then that island becomes full to capacity and some agents select a strategy at random on their current island instead.

have any number of agents on each island up to  $N(1+d)$  and for a given number of agents on each island, we can have any number playing each strategy, as there is a positive probability that nobody moves while a certain number mutate. It is therefore possible to go from any state to any other and the process is irreducible. QED.

**Definition 2.1:** *The set of states in the support of  $\lim_{e \rightarrow o} m(e)$  will be called the long-run equilibria.*

**Definition 2.2:** *A  $k$ -tree,  $h$ , defined on state space  $R$  (the set of recurrent communication classes), is a set of ordered pairs,  $(i \rightarrow j)$   $i, j \in R$ , such that each state  $x \neq k$  is the initial point of one arrow and from every state there is a path which leads to  $k$ .*

Let  $r_{ij}$  be the minimum number of mutations required to go from class  $i$  to  $j$ . We know that such a number exists because  $P^e$  is irreducible. The cost of a  $k$ -tree is  $\sum_{(i \rightarrow j) \in h} r_{ij}$ .

**Proposition 2.2:** *The long-run equilibria are the set of states in the recurrent communication class which has the lowest cost  $k$ -tree.*

For the proof the reader is referred to Young (1993). The intuition is clear. The long-run equilibria are the set of states in the recurrent communication class that is easiest to flow into from all other recurrent communication classes. Hence to find the long-run equilibria we need to characterise the recurrent communication classes and the costs  $r_{ij}$  of moving between them and then find the class that has the lowest cost  $k$ -tree.

### Recurrent communication classes.

One recurrent communication class is the set of all states where  $q_1 = q_2 = 0$ . The basin of attraction of this class is  $\{(q_1, q_2): q_1 \leq q^*, q_2 \leq q^*\}$ , since best replies will<sup>4</sup> lead both islands to coordinate on the risk-dominant equilibrium. In this class the system will move between states where  $q_1 = q_2 = 0$  and  $n_1 \in (N(1-d), N(1+d))$ , since agents move with a positive probability when they are indifferent and  $n_1$  must lie in this range due to the capacity constraint.

Now consider any initial condition with  $q_1 \geq q^*$  and  $q_2 \leq q^*$ . Best replies will move the system towards  $q_1 = 1$  and  $q_2 = 0$ . This will result in movement into island 1, since the higher payoff equilibrium is being played there. The system will eventually move to the equilibrium state  $(1, 0, N(1+d))$ . Similarly the set of states with  $q_1 \leq q^*$  and  $q_2 \geq q^*$  form the basin of attraction of the equilibrium  $(0, 1, N(1-d))$ . The final possibility is for both populations to coordinate on the payoff-dominant equilibrium. The basin of attraction for this class is  $\{(q_1, q_2): q_1 \geq q^*, q_2 \geq q^*\}$ , and the recurrent communication class is the set of all states with  $q_1 = q_2 = 1$  and  $n_1 \in (N(1-d), N(1+d))$ . The four recurrent communication classes are illustrated in figure 2.2.

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<sup>4</sup>When  $q_1 = q^*$  or  $q_2 = q^*$  the dynamics can go either way.

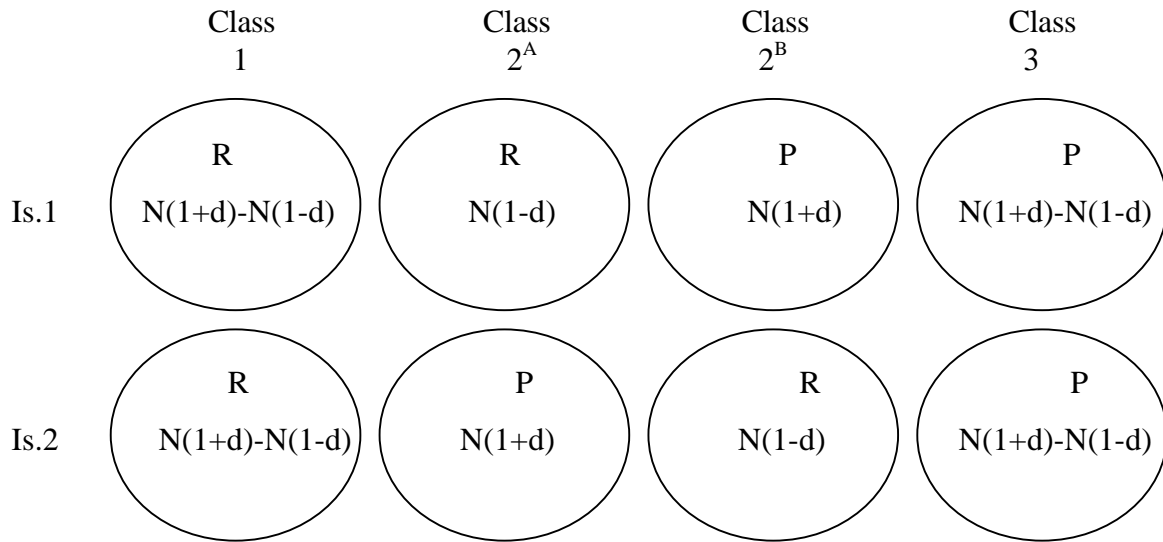


Figure 2.2. Recurrent communication classes:  
Row  $i$  of circles illustrate the equilibrium played on island  $i$  in each of the classes (risk-dominant, R or payoff-dominant, P), plus the range of values of  $n_i$  that are consistent with the class.

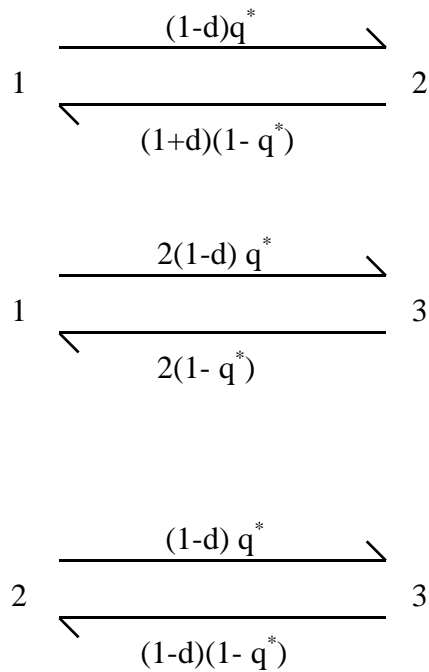


Figure 2.3  
Minimum costs of moving between recurrent communication classes.

### Minimum costs of moving between recurrent communication classes.

Consider the transition from class 1 to  $2^A$ . We want the minimum number of mutations required to get into the basin of attraction of class 2,  $\{(q_1, q_2): q_1 \geq q^*, q_2 \leq q^*\}$ , from a state in class 1,  $(0,0,n_1)$ . Hence we require a proportion  $q^*$  of island 1 to mutate. Now the less populated island 1 is, the lower the number of mutations required to achieve this. The minimum value of  $n_1$  is  $N(1-d)$  so the minimum number of mutations required is  $N(1-d) q^*$ . The dynamics will then move the system to the state  $(1,0,N(1+d))$ . The cost of moving back is  $N(1+d) (1-q^*)$  since we require the system to move back to a state where  $q_1 \leq q^*$  and island 1 is full to capacity.

A direct jump will not necessarily yield the minimum number of mutations. For example consider the transition from class 1 to 3. A direct jump from class 1 to 3 requires  $2Nq^*$  simultaneous mutations. However, it is easier to go from class 1 to 2 and then from 2 to 3 since this only requires  $2(1-d)Nq^*$  mutations. Hence the minimum number of mutations required to go from class 1 to 3 is  $2(1-d)Nq^*$ . All the minimum costs are given in figure 2.3.

**Lemma 1:** *To find the class which has the minimum cost k-tree it is sufficient to find the minimum cost trees between just three classes, ruling out either  $2^A$  or  $2^B$ .*

**Proof:** Let  $r_{ij}$  denote the minimum cost of the transition  $i \rightarrow j$ .

$$\text{Then } r_{i2^A} = r_{i2^B} \quad r_{2^A i} = r_{2^B i} \quad i \in (1, 2, 3) .$$

Let  $h$  be a minimum cost k-tree. Adjust the tree so that at least one of  $2^A$  or  $2^B$  have no predecessors without changing the cost. This is easy to do since  $i \rightarrow 2^A$  can be transferred to  $i \rightarrow 2^B$  (or vice versa) leaving  $2^A$  with no predecessors. We can split the adjusted k-tree into two parts, a minimum cost  $k'$ -tree defined on the vertices  $(1,2,3)$  and  $2^A$  added at minimum cost. It must be a minimum cost  $k'$ -tree because any adjustments which reduce the cost would also reduce the cost of the k-tree but we started with a minimum cost k-tree. Hence we can

find the minimum cost  $k$ -tree by first finding the minimum cost  $k'$ -tree and then adding a 2-state at minimum cost. This cost will be common to all  $k$ -trees and so does not need to be considered. QED

This leaves nine trees that we need to compare (3 for each communication class). These trees are illustrated in table 2.1.

1-trees	2-trees	3-trees
$A^1: 1 \leftarrow 2 \leftarrow 3$	$A^2: 2 \leftarrow 1 \leftarrow 3$	$A^3: 3 \leftarrow 2 \leftarrow 1$
$B^1: 1 \leftarrow 3 \leftarrow 2$	$B^2: 2 \leftarrow 3 \leftarrow 1$	$B^3: 3 \leftarrow 1 \leftarrow 2$
$C^1: 2 \rightarrow 1 \leftarrow 3$	$C^2: 1 \rightarrow 2 \leftarrow 3$	$C^3: 2 \rightarrow 3 \leftarrow 1$

Table 2.1.  
k-trees

**Proposition 2.3:** *The long-run equilibria are:*

*the set of states in class 1 if  $d < 2q^* - 1$ ,*  
*and states  $2^A$  and  $2^B$  if  $d > 2q^* - 1$ .*

**Proof:** From proposition 2.2, the long-run equilibria are the set of states in the recurrent communication class which has the lowest cost  $k$ -tree. It is a simple exercise to see that the lowest cost 1-tree is  $A^1: 1 \leftarrow 2 \leftarrow 3$ . The other two 1-trees include the transition  $1 \leftarrow 3$ , which has the same cost as  $A^1$  but also include a transition from class 2 at some cost. Similarly, the lowest cost 3-tree is  $A^3: 3 \leftarrow 2 \leftarrow 1$  as the other two 3-trees include the transition  $3 \leftarrow 1$ , which has the same cost as  $A^3$ . Finally, the lowest cost 2-tree is  $C^2: 1 \rightarrow 2 \leftarrow 3$ . The other two 2-trees include the transitions  $3 \rightarrow 1$  and  $1 \rightarrow 3$ . In each case the cost is reduced by going directly to class 2.

The only difference between the cost of  $C^2$  and  $A^3$  is in the transition between classes 2 and 3. Since  $r_{23} > r_{32}$  (as  $q^* > \frac{1}{2}$ ),  $C^2$  always has a lower cost. This leaves two candidates for minimum cost k-tree,  $A^1$  and  $C^2$ . The cost of  $A^1$  is less than the cost of  $C^2$  if  $r_{21} < r_{12}$ . Hence class 1 has the lowest cost k-tree if

$$(1+d)(1-q^*) < (1-d)q^* \Rightarrow d < 2q^* - 1 \quad (2.1)$$

If the inequality is reversed then class 2 has the minimum cost k-tree. QED.

The long-run equilibria are illustrated in figure 2.4. Hence the long-run equilibria are the set of states where everyone plays  $s_2$ , the risk-dominant strategy if  $d < 2q^* - 1$ . The critical value of  $d$  where class 1 becomes the long-run

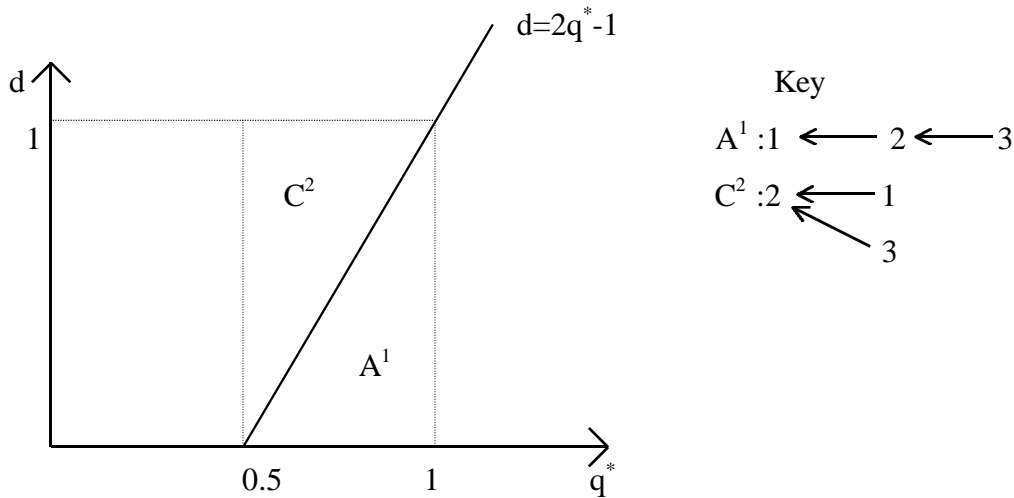


Figure 2.4  
Minimum cost j-trees.

equilibrium increases with the degree of risk-dominance. If  $d$  is above this critical value then class 2 has the minimum cost k-tree. The long-run equilibria are the two states where the two islands play different equilibria. In fact, from the symmetry of the cost structure, states  $2^A$  and  $2^B$  will each have a probability of one half in the limit-distribution. It is easy to see why higher



values of  $d$  upset the risk-dominant equilibrium. In class 2, the island playing the payoff-dominant equilibrium becomes more populated as  $d$  increases because in equilibrium it is full to capacity. The transition to class 1 therefore becomes more difficult.

By the same token the island playing the risk-dominant equilibrium in class 2 becomes smaller as  $d$  increases and so easier to convert. Hence the transition from class 2 to class 3 where both islands play the payoff-dominant equilibrium becomes easier. However, we never observe class 3 as the long-run equilibrium. The reason for this is that although the cost of class 3 is decreasing, the cost of class 2 is also decreasing and is always less. Consider the minimum costs of the transitions  $1 \rightarrow 2$  and  $3 \rightarrow 2$ . As  $d$  increases the smallest possible size of an island,  $(1-d)N$ , falls. In classes 1 and 3 the transition to a state where one island has a population size of  $((1-d)N)$  has zero cost. The transition to class 2 then only requires enough mutations on the small island. The minimum cost therefore falls as  $d$  increases.

In Anwar (1998), this model is extended to the case where there are three locations. As in the above analysis, the long-run equilibrium remains the state where all islands coordinate on the risk-dominant equilibrium when  $d$  is sufficiently small. However, if  $d$  is sufficiently high for any given level of risk dominance then the long-run equilibrium involves the different islands coordinating on different equilibria.

## **2.2 Inertia in strategy revision.**

The previous results rely on the assumption that each agent plays a best reply at their location. We now extend the model to the case where there is a positive probability that agents simply continue to use the strategy they used in the previous period. However, if they do revise their strategy they do so by playing a best reply. As before, there is a positive probability that they are given the chance to move islands and agents will then choose the location and strategy that would have maximised their expected payoff in the previous period,

as long as this does not involve moving to an island that is full to capacity. One can now think of the following story underlying these dynamics. At the end of each period the following events occur with a positive probability for each agent: 1) the agent observes nothing about the proportions using each strategy, 2) the agent only observes the proportions on his current island and 3) the agent observes the proportions on both islands. In the first case he simply continues to use the same strategy in the next period. In the second case he chooses a best reply on his current island. Finally, in the third case, he will want to move if a best reply on the other island yields a higher expected payoff than a best reply on his current island. If the island has spare capacity he will move and play the best reply. If it is full then he plays a best reply on his current island. The previous models look at the extreme case where the probability of the first event is zero. In this model the probability of each event is positive.

The state space  $S$ , is the same as in the case with no inertia. The above dynamics, however, give rise to a different transition matrix,  $P'$ . All other aspects of the model are the same. The perturbed transition matrix is given by

$$P_{ij}^{/e} = P_{ij}' (1 - e)^{2N} + \sum_{k=1}^{2N} c_{ijk} e^k (1 - e)^{2N-k}.$$

The same reasoning as before can be used to show that  $P^e$  is aperiodic and irreducible. Hence we can apply propositions 2.1 and 2.2 and find the long-run equilibria by finding the recurrent communication class that has the lowest cost  $k$ -tree. The recurrent communication classes are the same as in the model with no inertia and are illustrated in figure 2.2. However, the basins of attraction of the recurrent communication classes are now different. This changes the cost of moving between classes. In the model without inertia, the minimum cost of the transition  $2 \rightarrow 1$  is  $(1+d)(1-q^*)$ . We can now achieve this transition with fewer mutations because after a certain number of mutations on the efficient island, it will be optimal for agents to move and get a payoff of  $D$ . If there are  $(1-d)(1-q^*)$  mutations followed by movement, then

there is a positive probability that  $2Nd$  agents move and that all the agents that move were playing  $s_1$ , while nobody revises their strategy on the efficient island. Hence the proportion playing  $s_1$  will be  $(N(1+d)-2Nd-(1-d)(1-q^*)N)/(N(1+d)-2Nd)=q^*$ . However, we must ensure that it is optimal to move and this requires a proportion  $(1-q')$  of the efficient island to mutate, where  $q'$  satisfies  $Aq' + B(1-q') = D$  or  $q' = (D-B)/(A-B)$ .

Hence

$(1-q')(1+d)N$  mutations are required before anyone will move. Since the number of mutations must satisfy both of the above conditions, the minimum number of mutations required will be  $\max[(1-q')(1+d), (1-q^*)(1-d)]$ . The minimum costs of moving between recurrent communication classes are given in figure 2.10.

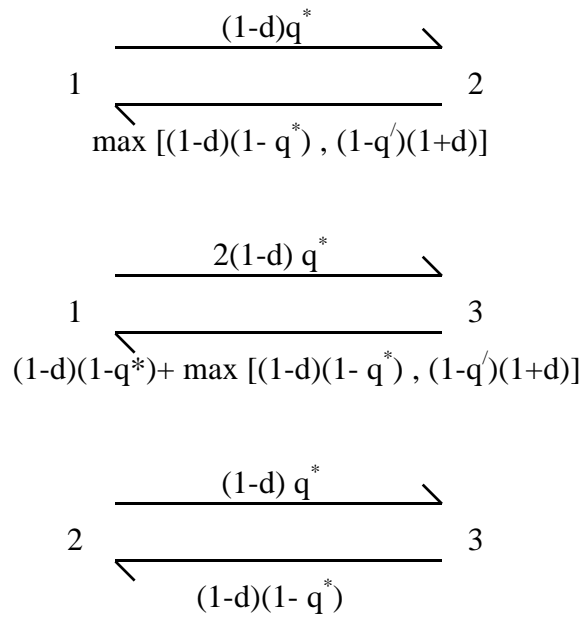


Figure 2.10

Minimum costs of moving between recurrent communication classes.

**Proposition 2.5:** *The long-run equilibria are:*

*all states in class 1 if  $d < f(q^*, q')$*

*and states  $2^A$  and  $2^B$  if  $d > f(q^*, q')$ ,*

$$\text{where } f(q^*, q') = \frac{q^* + q' - 1}{q^* + 1 - q'}.$$

**Proof.** From proposition 2.2 we know that we need to find the class with the lowest cost k-tree. Also to find the class which has the minimum cost k-tree it is sufficient to find the minimum cost trees between just three classes, ruling out either  $2^A$  or  $2^B$  (lemma 1). Of the nine trees (table 2.1), it is a simple exercise to see that the minimum cost tree is either  $1 \rightarrow 2 \leftarrow 3$  or  $1 \leftarrow 2 \leftarrow 3$ <sup>5</sup>. Hence the set of states in class 1 will be the long-run equilibria when

$$\max[(1-d)(1-q^*), (1-q')(1+d)] < (1-d)q^*.$$

This condition reduces to

$$(1-q')(1+d) < (1-d)q^* \Rightarrow d < \frac{q^* + q' - 1}{q^* + 1 - q'}.$$

If the inequality is reversed then class 2 has the minimum cost k-tree. Q.E.D.

The function  $f(q^*, q')$  is increasing in  $q^*$ . Apart from the transitions  $2 \rightarrow 1$  and  $3 \rightarrow 1$  the minimum costs of moving between recurrent communication classes are the same as the model with no inertia. Since  $c_{21}$  is smaller than in the earlier case, class 1 has a slightly larger range over which it is the long-run equilibrium. Otherwise the long-run equilibria are similar - class 1 if  $d$  is below some critical value and class 2 if it is above this value, where the critical value is increasing with  $q^*$ .

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<sup>5</sup> Any tree that includes the transition  $1 \leftarrow 3$  (or  $3 \leftarrow 1$ ) cannot be the minimum cost tree because  $1 \leftarrow 2 \leftarrow 3$  (or  $3 \leftarrow 2 \leftarrow 1$ ) has the same cost but includes no more transitions. Also  $3 \leftarrow 2 \leftarrow 1$  always has a higher cost than  $3 \rightarrow 2 \leftarrow 1$  because  $r_{32} < r_{23}$  for all values of  $d$ .

<sup>6</sup>  $(1-d)(1-q^*) < (1-d)q^*$  as  $q^* > 1/2$ .

## 2.3 Congestion

One of the interpretations for the capacity constraints given in the introduction is that they are imposed by local governance to limit congestion. A congestion effect can be introduced directly into the payoff matrix by subtracting some increasing function of the population size,  $f(\cdot)$ , from all the payoffs. We do not want the congestion effect to change the relative efficiency of the strategies, so we assume,

$$A-D > f(N(1+d)) - f(N(1-d))$$

With these additional effects, class 1 in the model of section 2.1 reduces to the state where there is a population of  $N$  on each island,  $(0,0,N)$ . Similarly class 3 reduces to the state,  $(1,1,N)$ . The four recurrent communication classes are 4 single states. However, the costs of moving between the classes is almost identical to the costs given in figure 2.3. The only significant difference is that to go from class 1 to class 2 requires  $(1-d)q^*N+1$  mutations. After one mutation, agents move away from the location where the mutation has occurred. This will leave  $(1-d)N$  on the island where the mutation occurred and a proportion  $q^*$  of those that remain must mutate to convert it. Hence condition 2.1 now becomes,

$$d < 2q^* - 1 + 1/N. \quad (2.2)$$

Hence for large  $N$  the condition is almost the same. The above result relies on one mutation followed by mass movement and this is because strategy revision was assumed to be instantaneous in section 2.1. In the case where there is inertia in strategy revision the reasoning and results of section 2.2 follow exactly. The congestion effect makes no difference whatsoever.

## 2.4 No capacity constraints.

We now consider the consequences of removing the capacity constraints altogether. To do this it is necessary to make some assumptions on what happens when an island becomes empty. We begin by looking at the case where there is no congestion effect. If we assume that the payoff of being alone at a location is less than  $D$  then there are 4 equilibrium states, all of which involve an empty island<sup>7</sup>. To ensure the perturbed process is irreducible, we only consider the case where agents choose a strategy and location at random when they mutate<sup>8</sup>. Only one mutation is required to move from an equilibrium where everyone plays  $s_2$  to one where everyone plays  $s_1$  but to move in the reverse direction requires  $2N(1-q^*)$  mutations. Clearly the long-run equilibria are the two states where one island is empty and the other one plays the efficient equilibrium. The empty location plays a coordinating role.

The story is slightly different if there is a congestion effect, as there are two further equilibria to consider,  $(1,1,N)$  and  $(0,0,N)$ . In the first both islands coordinate on the efficient equilibrium and in the second both coordinate on the risk-dominant one with the population evenly spread in each case. Call these states A and B respectively. We can simplify the analysis by allowing for a small degree of anticipation. When an island is empty an agent may move to it and play the efficient strategy anticipating the fact that others will follow onto the less congested island. Hence the states where one island is empty now form part of the basin of attraction of state A and there are only two equilibrium states, A and B. To move from state A to state B requires  $2N(1-q^*)$  mutations as we need both islands to convert to the risk-dominant equilibrium. To move from B to A only requires one mutation as agents will move away from the location where the mutation has occurred moving the system into the basin of

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<sup>7</sup>When the islands are playing different equilibria, then everyone will move to the efficient one. When they are playing the same equilibrium, then one island will eventually become empty without mutations as agents move with a positive probability when they are indifferent.

<sup>8</sup> Otherwise once an island becomes empty it remains empty.

attraction of the efficient equilibrium. Hence the only difference with the case where there is no congestion effect is that in equilibrium there is no empty location, the population is evenly spread.

### 3. Conclusions

Kandori *et al* and Young have developed techniques for characterising the limit of the stationary distribution when the mutation rate goes to zero. This allows us to address the question of equilibrium selection in the long-run when the mutation rate is very small. Using these techniques they show that the equilibrium selected is the risk-dominant one rather than the efficient one.

The results of the capacity constrained, local interaction model essentially show that the introduction of movement may upset the long-run equilibrium where everyone plays the risk-dominant strategy, but will not necessarily lead to a long-run equilibrium where everyone plays the efficient one. The alternatives are states in which some islands coordinate on the efficient equilibrium and others on the risk-dominant one. These equilibria are observed when there are binding capacity constraints. However, risk-dominance still has a role to play in the determination of the long-run equilibria. The important feature is the degree of risk-dominance which will determine how lax the capacity constraint needs to be before the long-run equilibria switch from purely risk-dominant to the mixture of equilibria. When the capacity constraints are relaxed altogether the long-run equilibria involve everyone playing the efficient strategy.

Hence when agents are able to move between locations, the state in which everyone plays the risk-dominant strategy is no longer the unique long-run equilibrium. However, results that show that movement leads to a unique long-run equilibrium where everyone plays the efficient strategy rely on spare capacity. In fact, with binding capacity constraints, the long-run equilibria will depend on the degree of risk-dominance and the strictness of the capacity constraints.

On a more practical level, the results show that imposing restrictions on movement hinders efficiency. The capacity constraints result in at least one location coordinating on the



inefficient equilibrium, and when the constraint is sufficiently strong, on both locations coordinating on the inefficient equilibrium. When the capacity constraints arise because of physical restrictions, there is potentially a gain to be made by expanding capacity at each location. The gain from efficiency would have to be balanced against the cost of expanding capacity.

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